A little *excursus* on Relativistic Simultaneity

And its paradoxes

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Chapter 1

Introduction

Galilean relativity, in 17th century, explained for the first time how velocities and spaces are not absolute; but in 1905, for the first time, a man called Albert Einstein proved that time was not absolute as well, pushing even further the meaning of measurements in physics and our understanding of the world. In this paper our goal will be to deeply explain the nature of events and their properties in relation to space and time, analyzing the consequences of Einsteins work.

1.1 Physics prerequisites

The following article is aimed at people who have already come into contact with the basics of special relativity. In particular, the following points must be fully comprehended in order to grasp the rest of the paper:

- *First Postulate of Special Relativity*: All (inertial) systems of reference are equivalent with respect to the formulation of the fundamental laws of physics.
- *Second Postulate of Special Relativity*: The speed of light, in empty space, is the same for all observers ($c = 299792458\frac{m}{s}$).
- Time dilation and length contraction equations:

$$\Delta t = \gamma \Delta \tau = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} \Delta \tau \tag{1.1}$$

$$\Delta \sigma = \gamma \Delta s = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} \Delta s \tag{1.2}$$

- Lorentz Equations:

$$x = \gamma \left(x' + vt' \right) \tag{1.3}$$

$$t = \gamma \left(t' + \frac{vx'}{c^2} \right) \tag{1.4}$$

$$t' = \gamma \left(t - \frac{vx}{c^2} \right) \tag{1.5}$$

or equivalently

$$\Delta x = \gamma \left(\Delta x' + v\Delta t' \right) \tag{1.6}$$

$$\Delta t = \gamma \left(\Delta t' + \frac{v\Delta x'}{c^2} \right) \tag{1.7}$$

Comparing equation 1.1 with 1.6, and 1.2 with 1.5: $\Delta t \rightarrow \Delta t$, $\Delta\tau \rightarrow \Delta t'$; $\Delta\sigma \rightarrow \Delta x$, $\Delta s \rightarrow \Delta x'$.

Chapter 2

Simultaneity's paradoxes

Definition: We say that two events E_1 and E_2, which occur respectively on the points P_1 and P_2, are simultaneous if the light beams emitted by the two points arrive at a midpoint M at the same time.

In fact, referring to the second postulate of special relativity, we know that the speed of light, in empty space, is the same for all observers. Therefore:

$$\begin{cases} c = \dfrac{\overline{P_1M}}{\Delta t_1} \\ c = \dfrac{\overline{P_2M}}{\Delta t_2} \\ \overline{P_1M} = \overline{P_2M} \end{cases} \Rightarrow \begin{cases} \dfrac{\overline{P_1M}}{\Delta t_1} = \dfrac{\overline{P_2M}}{\Delta t_2} \\ \overline{P_1M} = \overline{P_2M} = d_M \end{cases} \Rightarrow \frac{d_M}{\Delta t_1} = \frac{d_M x}{\Delta t_2} \Rightarrow \Delta t_1 = \Delta t_2 \tag{2.1}$$

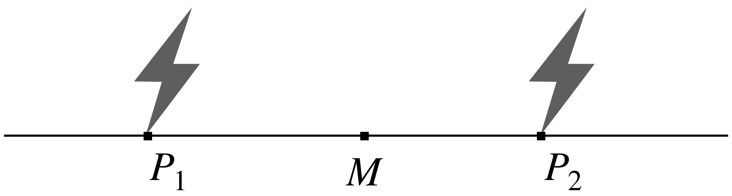

Figure 2.1

2.1 Einstein's train paradox

Einstein's train paradox is the most famous example of how simultaneity of events in general relativity becomes relative. In this paradox we imagine two light bolts hitting two different points P_1 and P_2 in the train track at the same instant, as shown in figure 2.2.

We also imagine two observers, one watching the train from the ground at a given point M and another one sitting on the train at a given point M', assuming that, in the track's reference, they are initially equidistant from the points P_1 and P_2.

Observer M is hit by the light of the light bolts at the same time, but observer M first perceives the light source it is going against by moving.

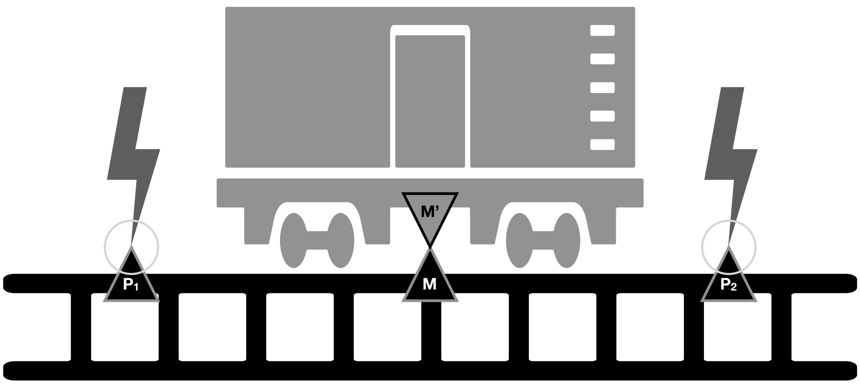

Figure 2.2

This paradox though, may give the idea that the events are relative only by an optical illusion, but again, the events observed by M' not only appear optically different, for that observer they actually happened at different times.

2.1.1 Train track's reference frame

As we already said, in the train track reference the two light bolts fall at the same time in the instant, which we will call t_0, when the points M and M' are overlapped and equidistant from the points P_1 and P_2, so that $\overline{P_1M} = \overline{P_2M} = \overline{P_1M'} = \overline{P_2M'}$, as shown in figure 2.2.

The light emitted by the two light bolts starts travelling heading towards M and M'. While M stays still in his reference, M' instead travels ahead the light emitted by the P_1's light bolt and away from the light emitted by the P_2's light bolt.

This results in the observer M' seeing before the P_2's light bolt and after the P_1's light bolt, as shown in figures 2.3, 2.5; the observer M instead, as we expect from equation 2.1, sees the two light bolts at the same time (figure 2.4).

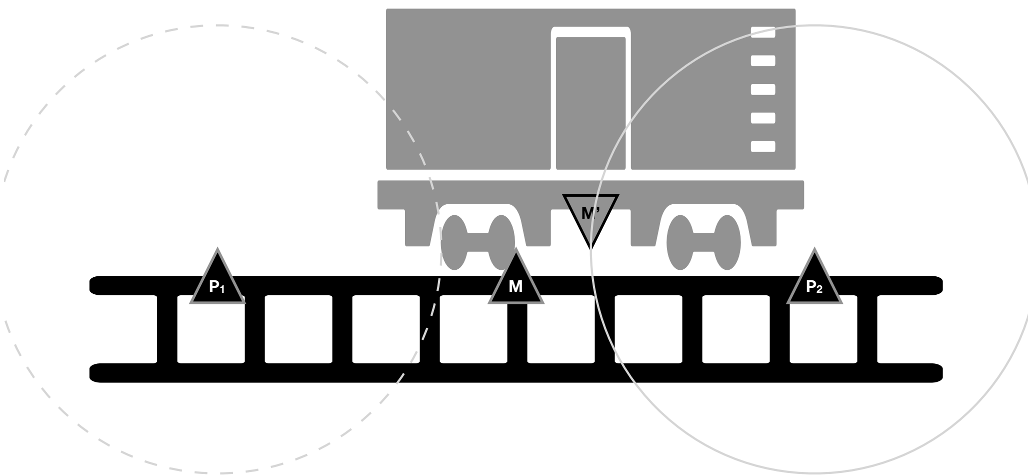

Figure 2.3

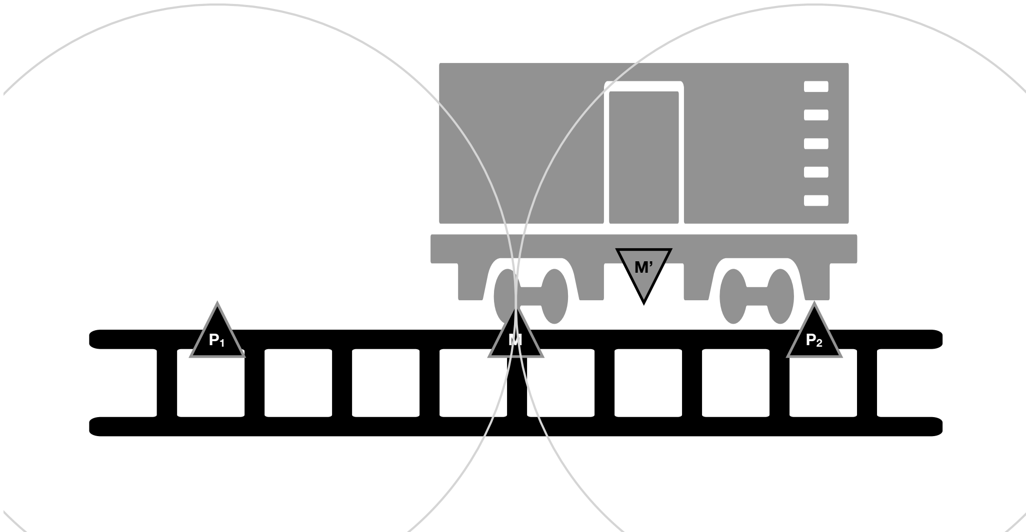

Figure 2.4

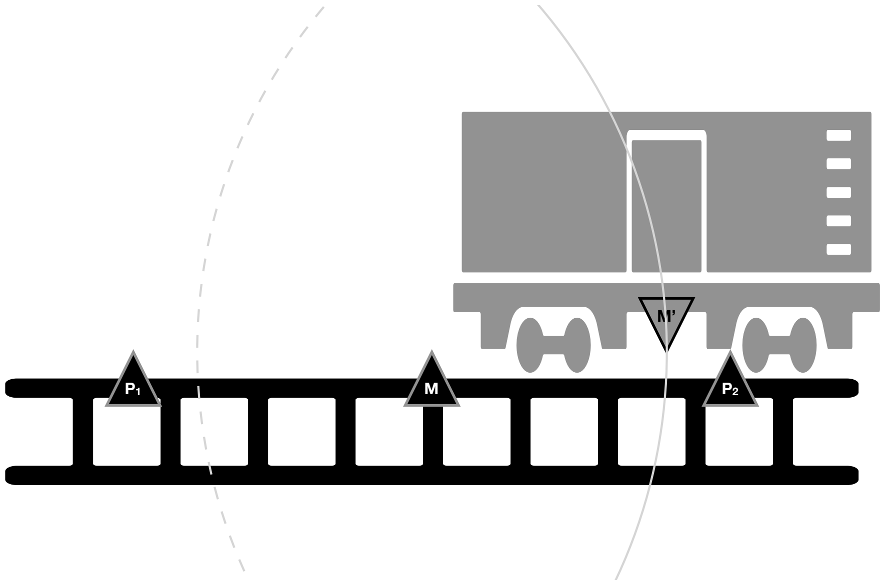

Figure 2.5

2.1.2 Train's reference frame

Now we imagine the same situation in the train's reference frame.

In this reference, the train, and so the observer M', stays still and the rails, and so the observer M, travel at a relative speed v to the left, as shown in figure 2.6.

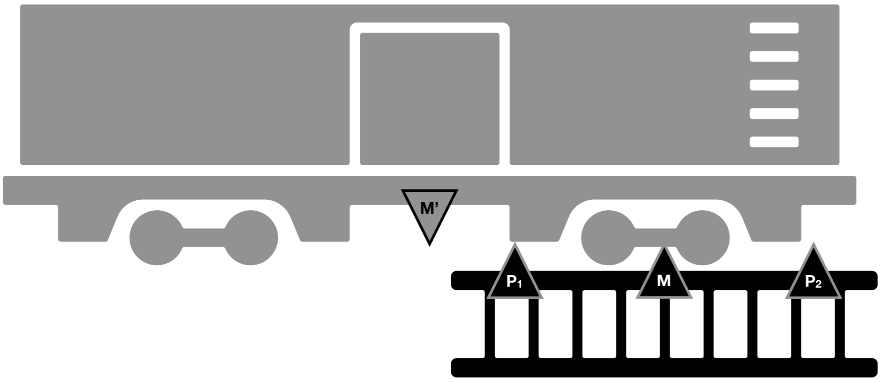

Figure 2.6

At a certain instant, which we are going to call $t'_{0,2}$, the P_2's light bolt happens in the train's reference frame and its light starts spreading towards M and M' (figure 2.7).

Due to the *Second Postulate of Special Relativity* the points P_2 and P_1 will be the center of their respective rays of light's propagation only in the instant they emit them.

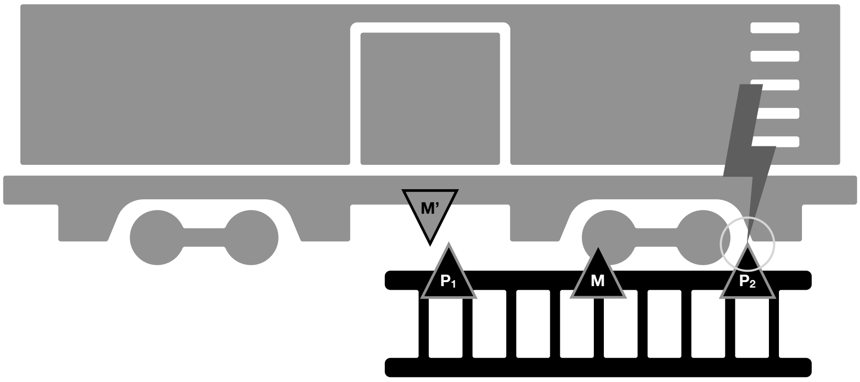

Figure 2.7

Now the P_2's light bolt's light will spread until it reaches M' (figure 2.8).

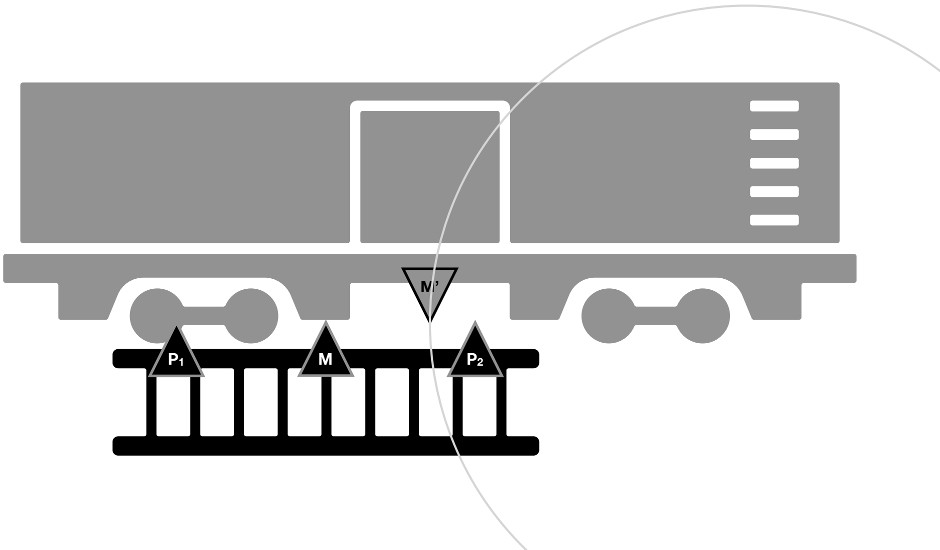

Figure 2.8

At a certain instant, which we are going to call $t'_{0,1}$, the P'_1's light bolt happens in the train's reference frame (figure 2.9).

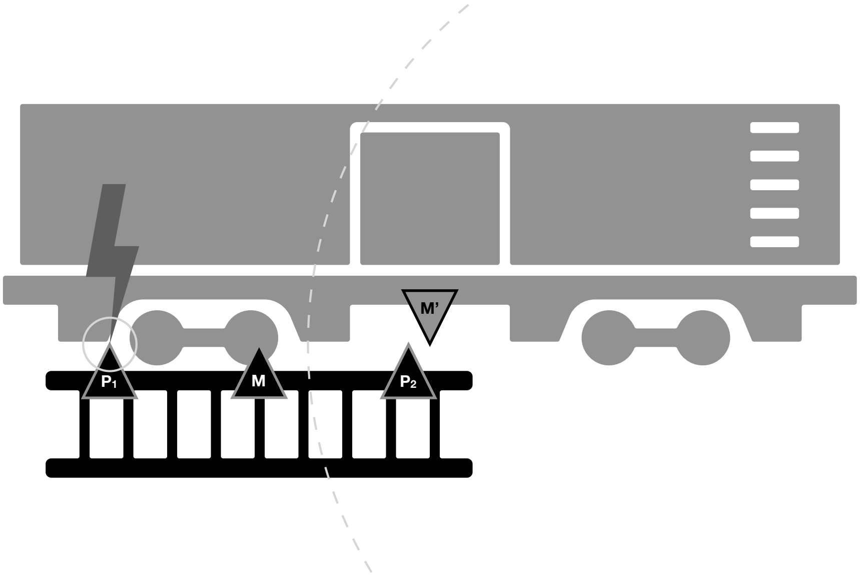

Figure 2.9

Then the P_2's light bolt's and the P'_1s light bolt's lights will reach M together.

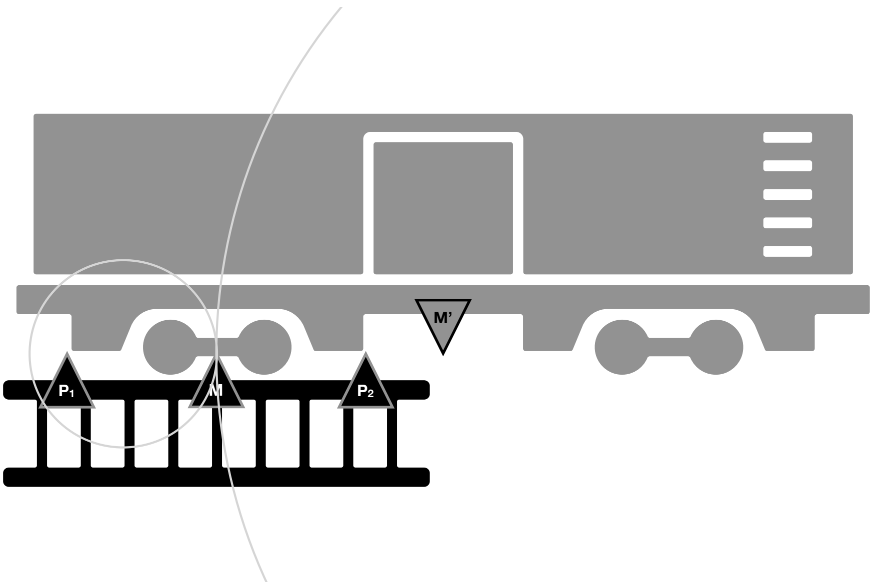

Figure 2.10

Finally the P_1's light bolt's light will reach M'.

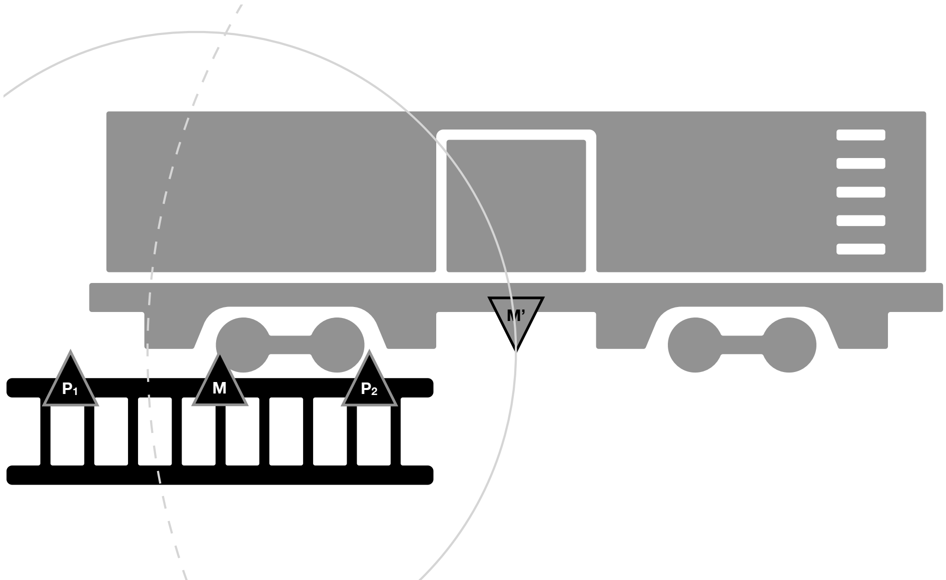

Figure 2.11

So we can conclude that also in the train's reference frame the two events won't be seen simultaneously by the observer M', while they will be seen simultaneously by the observer M.

If this statement couldn't be true it would mean that non only the timing of the events, in different reference frames, is different but also that the existence of the single event itself would be different, meaning that an event which, for example, exists for the train doesn't exist for the rails, and that is impossible.

The order of these events may variate in relation with the relative speed and the length of the train but the event shown in figure 2.10 will always happen between the events shown in figure 2.8 and figure 2.11.

2.1.3 Determining the time interval measured by the train track

Let's now calculate the time interval Δt that exists between the two events in the **train's track reference frame**.

We set in the train's track reference a system of coordinates positive towards right with his origin in M.

In the first place it will be necessary to determine the equation of motion, in the reference of the train track, of the ray of light coming from P_2:

$$X_2(t) = -ct + d$$

The equation of motion, in the reference of the train track, of the ray of light coming from P_1 is:

$$X_2(t) = ct - d$$

Where d is the distance $\overline{P_2M} = \overline{P_1M} =$ measured in the train track reference frame.

Finally the equation of motion, in the reference of the train track, of the point M' is:

$$X_{M'}(t) = vt$$

Therefore, the ray of light coming from P_2 reaches M' if:

$$-ct_2 + d = vt_2 \Longleftrightarrow t_2 = \frac{d}{c+v}$$

And the ray of light coming from P_1 reaches M' if:

$$ct_1 - d = vt_1 \Longleftrightarrow t_1 = \frac{d}{c-v}$$

Now let's calculate the interval $\Delta t = t_1 - t_2$ that passes between the instant t_2, when the P_2's light bolt's light reaches M' in the train track's reference frame, and the instant t_1, when the P_1's light bolt's light reaches M' in the train track's reference frame.

$$\Delta t = \frac{d}{c-v} - \frac{d}{c+v} = \frac{d(c+v) - d(c-v)}{c^2 - v^2} = \frac{dc + dv - dc + dv}{c^2 - v^2}$$

$$= \frac{2dv}{c^2(1 - \frac{v^2}{c^2})} = \gamma^2 \frac{2dv}{c^2}$$

So in the train track's reference frame the two events happen in the train with a Δt:

$$\Delta t = \gamma^2 \frac{2dv}{c^2}$$

2.1.4 Determining the time interval measured by the train

Let's consider now the same situation in the **train's reference frame**.

Using the Lorentz Equations for time (1.5) we can determine the previously named $t'_{0,2}$ and $t'_{0,1}$ which are respectively the instants when the P_2's light bolt and the P_1's light bolt happen in the train's reference frame:

$$t'_{0,2} = \gamma \left(t_{0,2} - \frac{x_2 v}{c^2} \right)$$

$$t'_{0,1} = \gamma \left(t_{0,1} - \frac{x_1 v}{c^2} \right)$$

We know that both light bolts happen at the same time in the train track's reference frame, so $t_{0,2} = t_{0,1} = 0$.

We also know that the x_2 and x_1 coordinates of both the light bolts are respectively equal to $+d$ and $-d$, in the train track's reference frame.

So we can write:

$$t'_{0,2} = \gamma \left(0 - \frac{dv}{c^2} \right) \iff t'_{0,2} = -\gamma \frac{dv}{c^2}$$

$$t'_{0,1} = \gamma\left(0 - \frac{-dv}{c^2}\right) \iff t'_{0,1} = \gamma\frac{dv}{c^2}$$

Now we can calculate the interval $\Delta t' = t'_{0,1} - t'_{0,2}$, in the train's reference frame, between the two events.

$$\Delta t' = \gamma\frac{dv}{c^2} - \left(-\gamma\frac{dv}{c^2}\right) = \gamma\frac{2dv}{c^2}$$

Now we start noticing some similarities between Δt and $\Delta t'$.

We have to remember that $\Delta t'$ is the interval between the two events in the train's reference frame and it is equal to, due to the equation 2.1, the interval between the instants in which P_2's light bolt's light and P_1's light bolt's light reach the observer M', also measured in the train's reference frame; Δt instead is the same interval but measured in the train track's reference frame.

So we can state that $\Delta t'$ is a proper time while Δt is a non proper time.

Given that we can use the equation of time dilatation (1.1):

$$\Delta t = \gamma\Delta t' \iff \gamma^2\frac{2dv}{c^2} = \gamma\left(\gamma\frac{2dv}{c^2}\right) \iff 1 = 1$$

This means that actually two events that are simultaneously for a reference frame are not simultaneous for another reference frame which is moving relatively to the first one and that the interval between the light emitted by the two events reaching the moving observer measured in the first reference frame is actually the dilated time interval, measured in the moving reference frame, with which the two events happen.

2.2 Einstein's train paradox, light sensors' variation

At this point we will provide an equivalent version of the previous paradox, useful for understanding the phenomenon. Instead of thunderbolts we will use two light sensors, located on the train tracks at points A and B. When the front wheel of the train passes over point B and the rear wheel passes over point A, the light sensors emit a beam of light which is directed towards point M, equidistant from both ends. Let's see what happens in the two different reference frames:

- The train track reference frame;
- The train reference frame.

They are presented on the next page.

2.2.1 Train track's reference frame

We are now in the train track's reference frame.

The train moves to the right with a relative speed v. Its wheels, which we will call A' and B' respectively, are approaching points A and B, as shown in figure 2.12.

It is important to remember that, in the track reference, $\overline{AM} = \overline{BM} = \overline{A'M'} = \overline{B'M'}$.

Where M' is the midpoint of $\overline{A'B'}$.

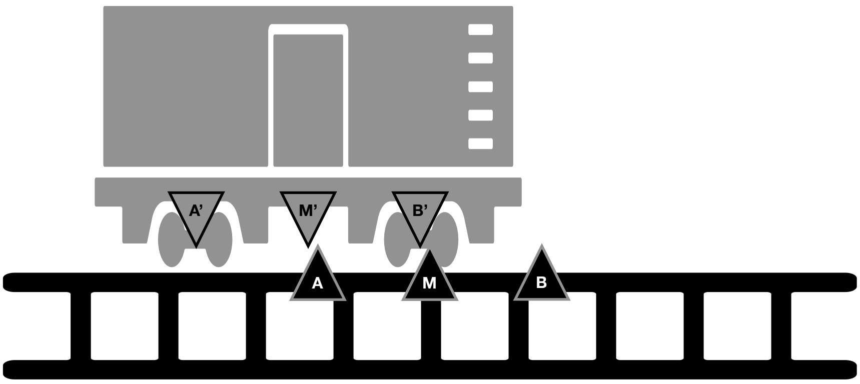

Figure 2.12

The train passes over points A and B and, as stated above, the wheels of the train, or points A' and B', will coincide for an instant with points A and B of the tracks.

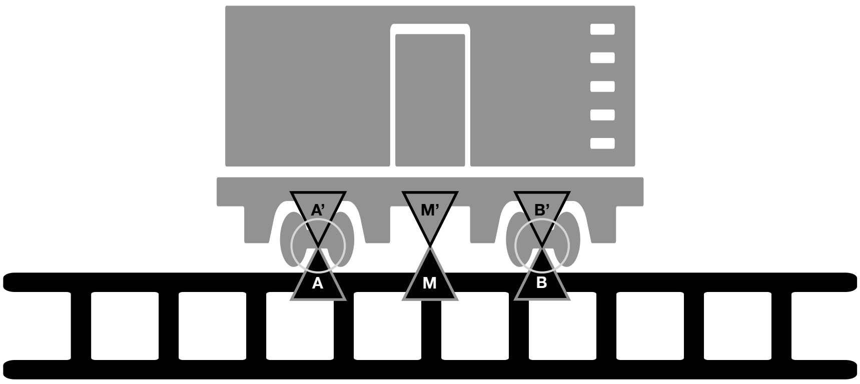

Figure 2.13

In this instant the light sensors emit their signal directed towards the point M of the rails, as shown in figures 2.13; 2.14; 2.15.

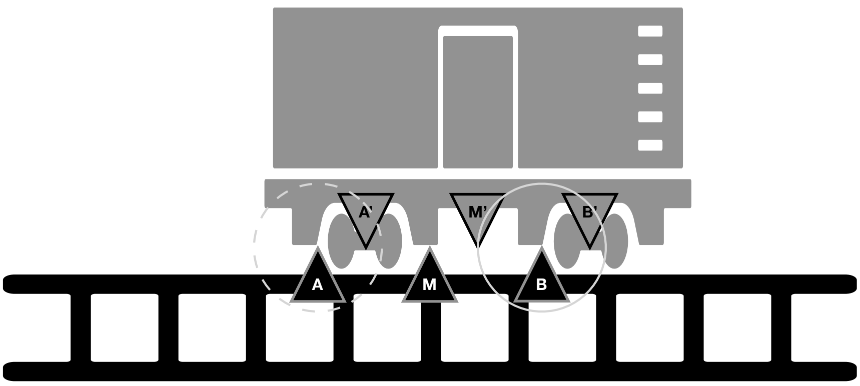

Figure 2.14

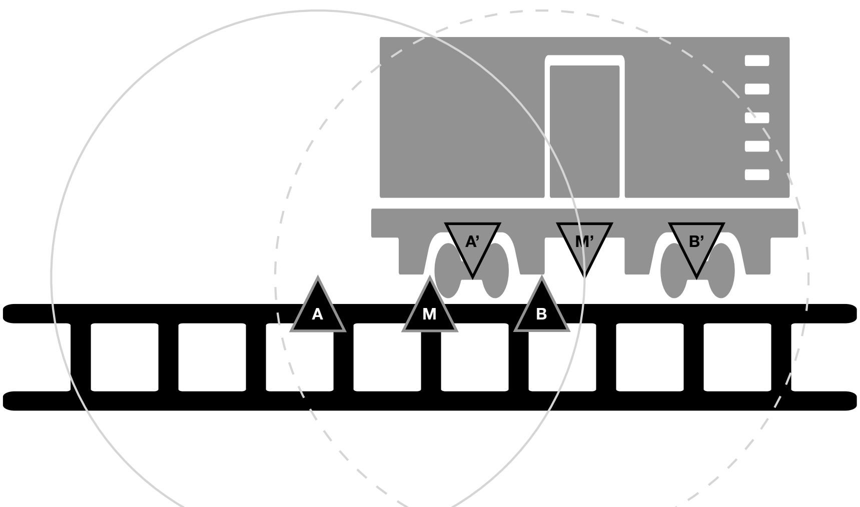

Figure 2.15

As it is possible to see, the two events, that are the emission of the two light signals, are simultaneous in the reference system of the train tracks. In fact, the space that the light will have to travel will be the same in both cases and therefore the time taken to travel this space will also be the same. Ultimately the two events satisfy the definition of simultaneity presented in 2.1.

2.2.2 Train's reference frame

Now it is necessary to imagine the same situation in another reference frame, the train's reference frame.

In this reference the train is stationary, and it is the rails that move at relative speed v to the left, as shown in figure 2.16

Furthermore, the rails themselves will be shortened, due to the equation of space contraction presented before (1.2).

Therefore the equality $\overline{AM} = \overline{BM} = \overline{A'M'} = \overline{B'M'}$ will no longer be true.

There will not be an instant where points A and B will coincide with points A' and B' respectively.

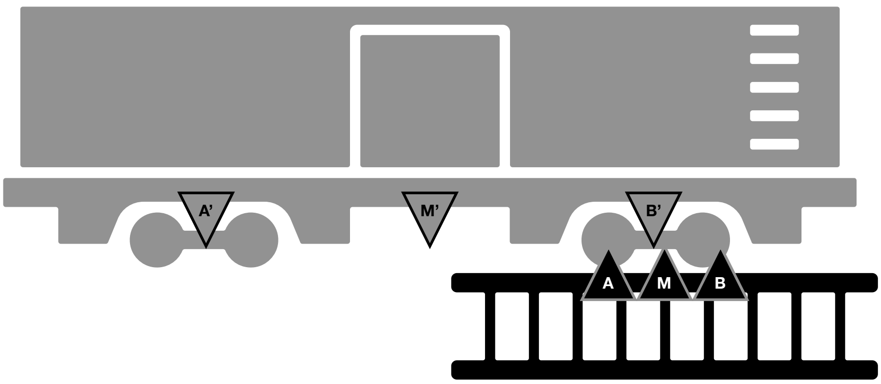

Figure 2.16

Instead, initially it will be the front wheel of the train (B') to pass over point B, triggering the light sensor which will send the signal to the center M', as shown in figure 2.17.

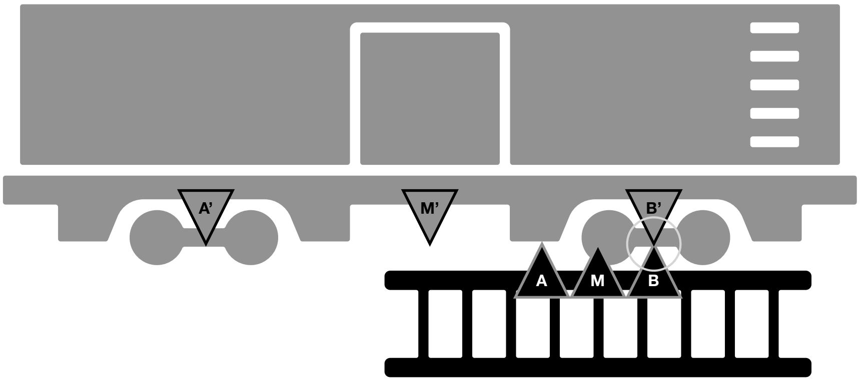

Figure 2.17

The signal will be propagated until it reaches the center (figure 2.18)

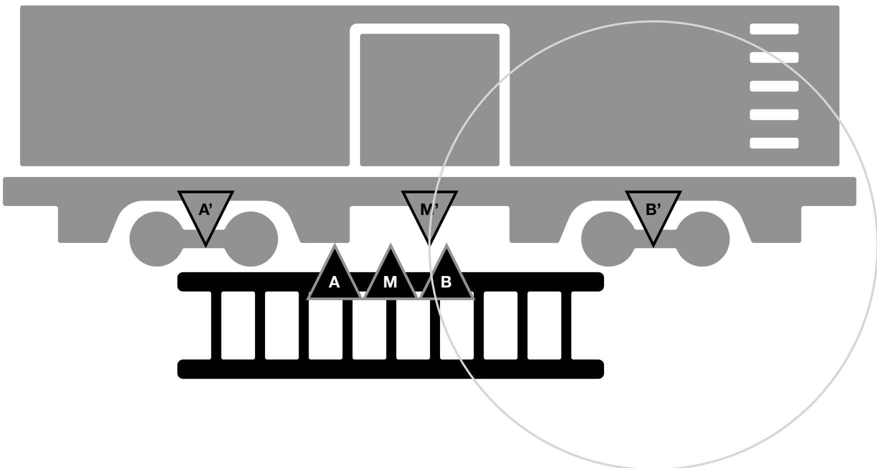

Figure 2.18

After a certain time interval Δt also the rear wheel (A') will pass over the point A, triggering the light sensor which will send the signal to the center M', as shown in figure 2.19.

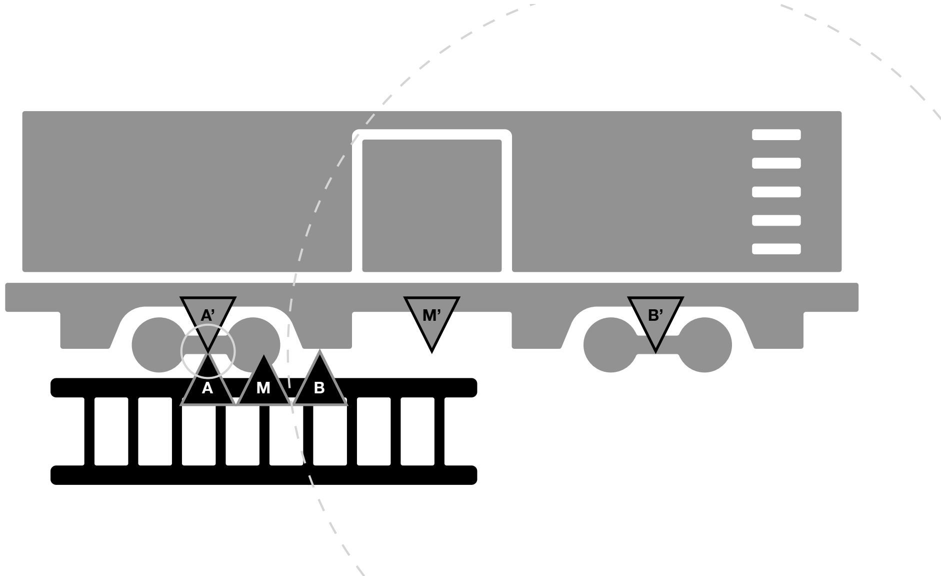

Figure 2.19

The signal will be propagated until it reaches the center (figures 2.20, 2.21)

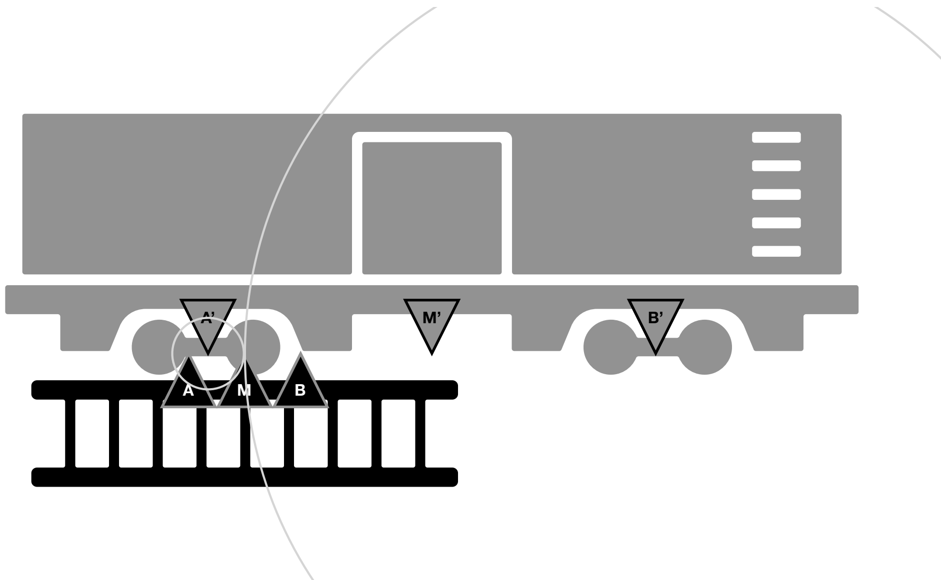

Figure 2.20

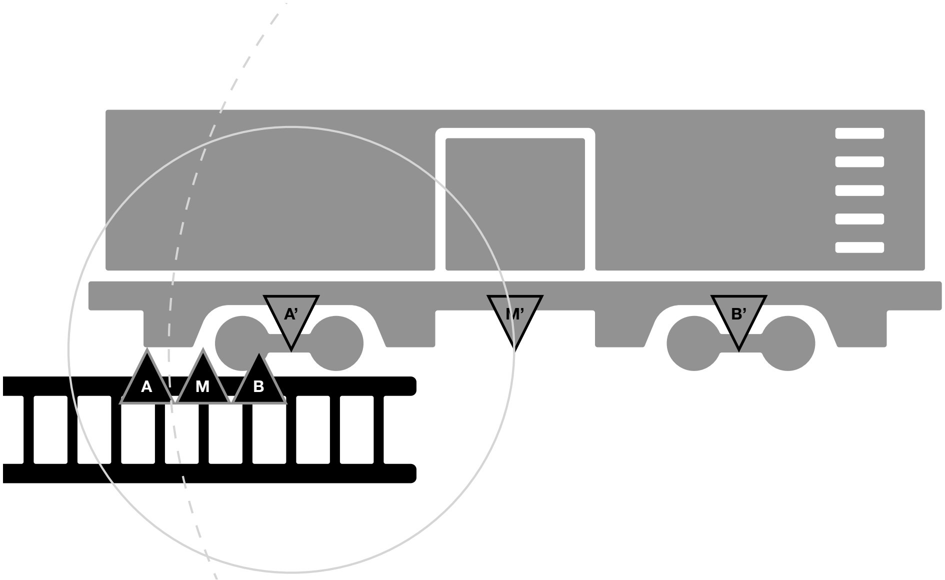

Figure 2.21

The two events do not respect the definition of simultaneity (2.1), as the two signals transmitted by the light sensors arrive at different instants.

Therefore the two events, in the train's reference frame, are not simultaneous.

2.2.3 Determining the time interval measured by the train track

Let's now calculate the time interval Δt that exists between the two events in the **train track's reference frame**.

In the first place it will be necessary to determine the equation of motion, in the reference of the train track, of the ray of light coming from A':

$$X_{A'}(t) = ct$$

The equation of motion, in the reference of the train track, of the ray of light coming from B' is:

$$X_{B'}(t) = -ct + l$$

Where "l" is the length of the train in the train track's reference frame, and $l\gamma = L$ is the length of the train in the train's reference frame.

Finally the equation of motion, in the reference of the train track, of the point M' is:

$$X_{M'}(t) = vt + \frac{l}{2}$$

Therefore, the ray of light coming from A' reaches M' if:

$$ct = vt + \frac{l}{2} \Longleftrightarrow t = \frac{l}{2(c-v)}$$

And the ray of light coming from B' reaches M' if:

$$-ct + l = vt + \frac{l}{2} \Longleftrightarrow t = \frac{l}{2(c+v)}$$

We will call the first time obtained t_1 and the second time t_2.
If the two events were simultaneous we should be able to prove that $t_1 = t_2$ and therefore $t_1 - t_2 = 0$.
On the other hand, following the previous arguments, we expect to find a time interval $\Delta t_1 \neq 0$.
Let's calculate it.

$$\begin{aligned} t_1 - t_2 &= \frac{l}{2(c-v)} - \frac{l}{2(c+v)} = \frac{+lc + lv - lc + lv}{2(c^2 - v^2)} \\ &= \frac{2lv}{2(c^2 - v^2)} = \frac{lv}{c^2(1 - (\frac{v}{c})^2)} = \gamma^2 l \frac{v}{c^2} \end{aligned}$$

We have shown that in the train track's reference frame the two events are not simultaneous and that the time interval between the first event and the second is exactly equal to:

$$\Delta t = \gamma^2 l \frac{v}{c^2}$$

2.2.4 Determining the time interval measured by the train

Now we may consider the same situation in the **train's reference frame**.

In this reference we have that $\overline{AB} = \frac{l}{\gamma} = \frac{L}{\gamma^2}$ and $\overline{A'B'} = L$

Let's consider as the starting instant the figure 2.17, in witch we fix a system of coordinates positive towards right with his origin in A'.

We proceed by writing the equation of the motion of the points B, A, B' and A' in the train's reference frame:

$$X_B(t') = -vt' + L$$

$$X_A(t') = -vt' + \left(L - \frac{L}{\gamma^2}\right)$$

$$X_{B'}(t') = L$$

$$X_{A'}(t') = 0$$

The event of the light being emitted by the B's light sensor happens when:

$$X_B(t'_B) = X_{B'}(t'_B)$$

Similarly the event of the light being emitted by the A's light sensor happens when:

$$X_A(t'_A) = X_{A'}(t'_A)$$

Now we can solve those equations (2.2.4; 2.2.4):

$$-vt'_B + L = L \Longleftrightarrow t'_B = 0$$

$$-vt'_A + \left(L - \frac{L}{\gamma^2}\right) = 0 \Longleftrightarrow t'_A = \frac{L}{v} - \frac{L}{v\gamma^2} = \frac{L}{v} - \frac{L}{v}\left(1 - \frac{v^2}{c^2}\right) = \frac{vL}{c^2}$$

So the interval between the two events, which is also, due to the equation 2.1, the interval between the instants in which B's light sensor's light and A's light sensor's light reach the observer M', measured in the train's reference frame equals to:

$$\Delta t' = t'_A - t'_B = \frac{vL}{c^2} - 0 = \frac{vL}{c^2}$$

So we can state that $\Delta t'$ is a proper time while Δt is a non proper time.

Given that we can use the equation of time dilatation (1.1):

$$\Delta t = \gamma \Delta t' \Longleftrightarrow \gamma^2 l \frac{v}{c^2} = \gamma \frac{vL}{c^2}$$

but, by definition, $l\gamma = L$

$$\gamma \frac{vL}{c^2} = \gamma \frac{vL}{c^2} \Longleftrightarrow 1 = 1$$

By solving this we demonstrated, again, that actually two events that are simultaneously for a reference frame are not simultaneous for another reference frame which is moving relatively to the first one and that the interval between the light emitted by the two events reaching the moving observer measured in the first reference frame is actually the dilated time interval, measured in the moving reference frame, with which the two events happen.

2.3 Car and garage paradox

The *car and garage paradox*, also known as *ladder paradox*, is another common example of the relativity of events, this model is perfectly equivalent to the previous variation of the Einstein's train paradox, but in this case we consider only the triggering events of the two light sensors. In this paradox we imagine a garage of proper length $\overline{AB}$, who sees a car moving at a speed such that the car's proper length $\overline{A'B'}$ is contracted to exactly $\overline{AB}$, since it moves at a relativistic speed towards the garage (figure 2.22).

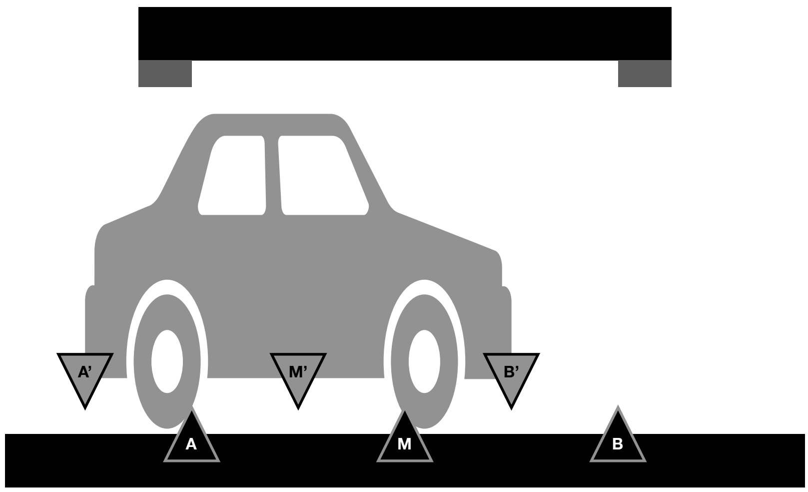

Figure 2.22

For the car though, the garage is shorter than the car, as in its reference frame $\overline{A'B'} > \overline{AB}/\gamma$ (figure 2.23).

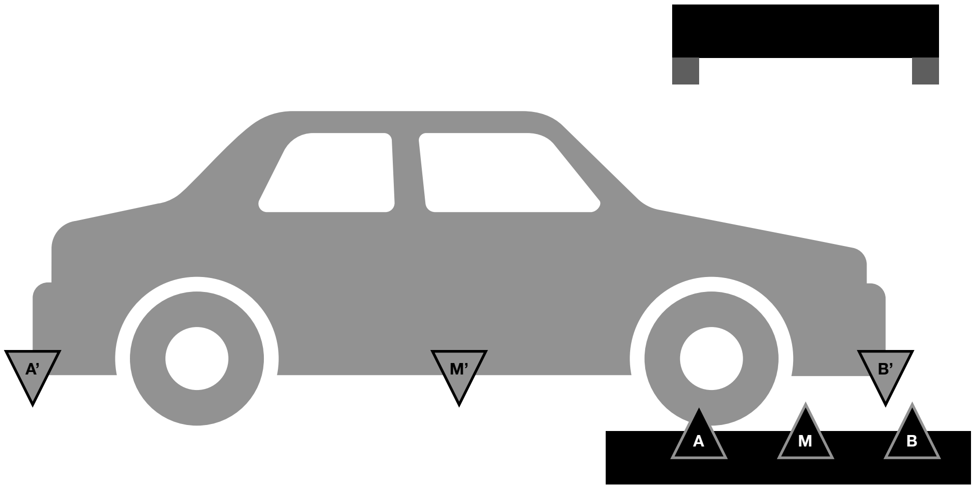

Figure 2.23

2.3.1 The nature of the paradox

The experiment consists on closing for an instant the front gate - equivalent to the left light sensor - and the back gate - equivalent to the right light sensor - of the garage at the same time, trying to close the car inside: from the garage reference frame by hypothesis $\overline{AB} = \overline{A'B'}$, so that the car perfectly fits inside the garage (figure 2.24).

But as described before, on the car reference frame we would see the garage smaller than the car itself: in this case we would expect the car being hit by both gates, contradicting the events on the other frame leading to a reliability issue (figure 2.25).

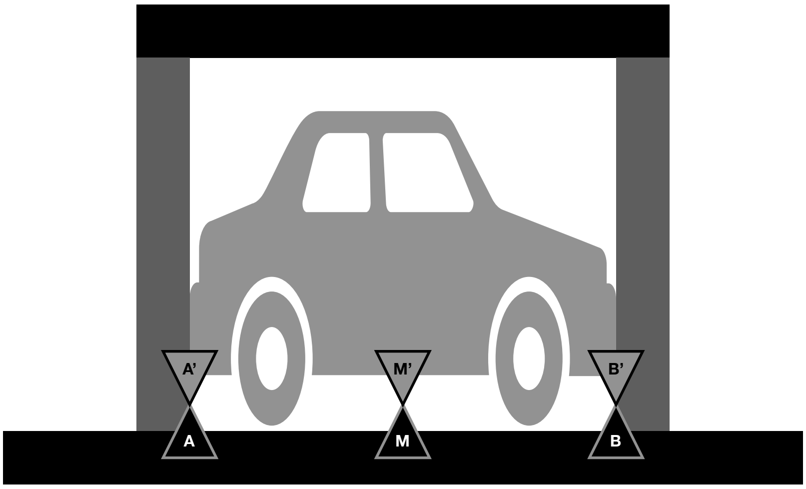

Figure 2.24

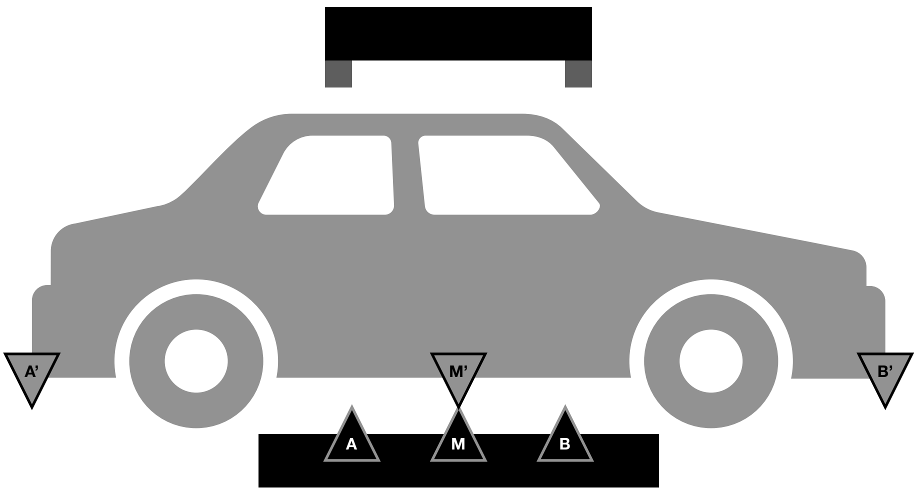

Figure 2.25

2.3.2 Solving the paradox

We must first understand that this situation is based on special relativity's contraction of distances, which is, in turn, the other side of the coin of the contraction of times. The problem in fact resides precisely in the transition from a reference frame to the other, and, since every reference frame has its own associated coordinate system, from a coordinate system to the other.

When we switch between two coordinate systems we use *Lorentz Equations*, and in this case we already know by hypothesis how spaces contraction are transformed between the two, but we still don't know how time contractions are. In particular, we can't just assume by heart that the two-time coordinates of the events of the doors closing change in the same way; we must consider the possibility that they could change in different ways, leading to perceived non-simultaneity of the two, in the car's reference frame.

The paradox is then solved by realising that for both observers, at a certain point in time, the head of the car will coincide with the front door, and the tail of the car will coincide with the back door.
The car reference perceives the events of the closing gates in different times: first, the front gate closes and opens instantly (figure 2.26), then the car continues to travel until its tail surpasses the back gate, which closes only after it is left behind (figure 2.27).

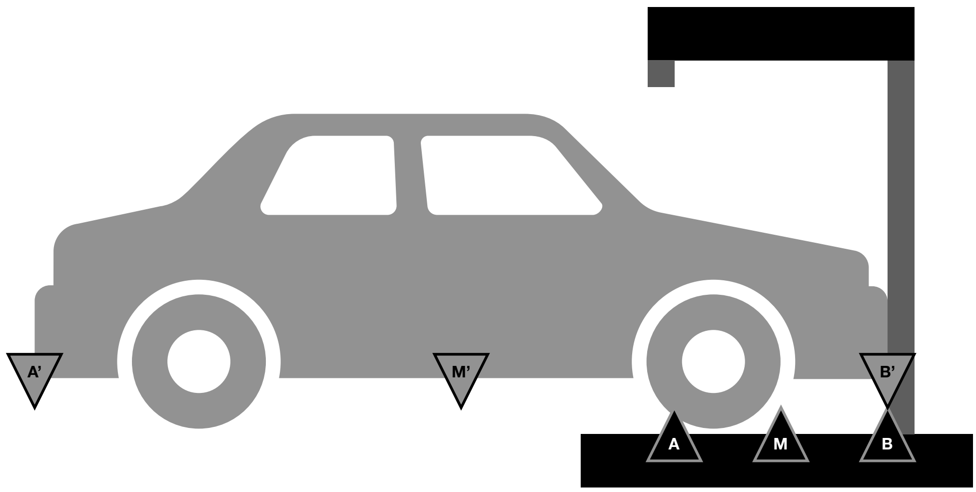

Figure 2.26

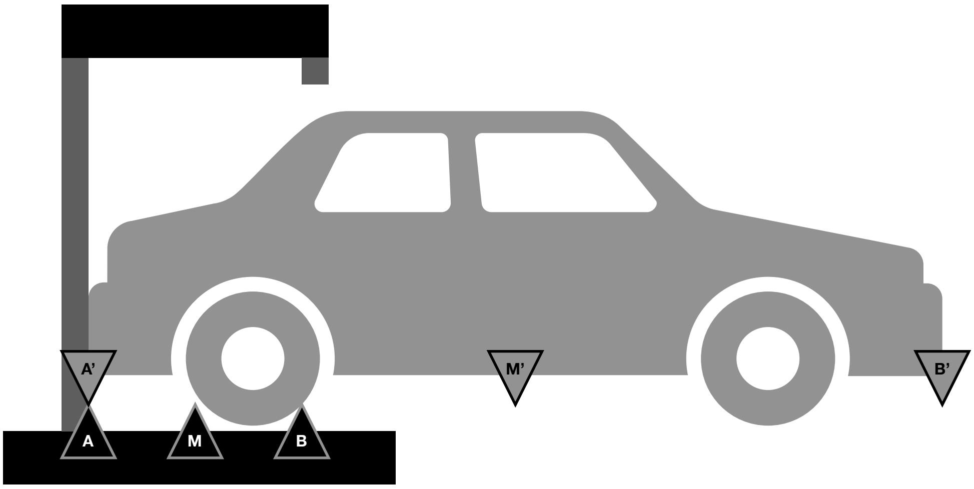

Figure 2.27

2.3.3 Determining the time interval measured by the car

To determine the time variation $\Delta t'$ between the events shown in figures 2.26 and 2.27 we can just use the Lorenz transformations (equation 1.5) to find the time of the two events t'_1 (figure 2.26) and t'_2 (figure 2.27) and subtract them. We will first assume t_1 and t_2 (the corresponding time coordinates in the garage's frame) as 0 and consider x_1 and x_2 as the corresponding spacial coordinates of the back gate (points A and B in figure), with the origins of the coordinate system O set respectively on point M. We will also assume that the two coordinate systems are both oriented as the velocity vector of the car.

$$t'_1 = \gamma\left(t_1 - \frac{vx_1}{c^2}\right) = -\gamma\frac{vx_1}{c^2} = -\gamma\frac{v\left(-\frac{l}{2}\right)}{c^2} = \gamma\frac{vl}{2c^2}$$

$$t'_2 = \gamma\left(t_2 - \frac{vx_2}{c^2}\right) = -\gamma\frac{vx_2}{c^2} = -\gamma\frac{v\left(\frac{l}{2}\right)}{c^2} = -\gamma\frac{vl}{2c^2}$$

where l, as in 2.1.3, is the proper length $\overline{AB}$ of the garage.
We can now conclude that

$$\Delta t' = |t'_2 - t'_1| = \left|-\gamma\frac{vl}{2c^2} - \gamma\frac{vl}{2c^2}\right| = \gamma\frac{vl}{c^2}$$

With this final equation we have finally proved that the events are perceived by the two observers as stretched and staggered with direct proportionality to the relative velocity v and the distance between the event and the observer. In other words, for the moving observer, the more an event is distant from him, and the faster he goes, the more his time will be staggered from the other stationary observer.

Chapter 3

Chronological order and causality

3.1 Chronological order

Under the same assumptions of section 2.2 we will now suppose that - when the front wheel of the train passes over point B and the rear one passes over A - the light sensor in B emits a beam of light with a delay quantified in Δt with respect to A, as shown in the following figures.

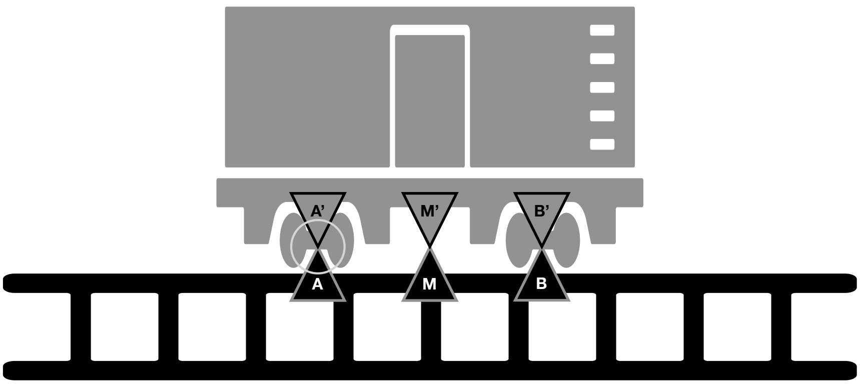

Figure 3.1

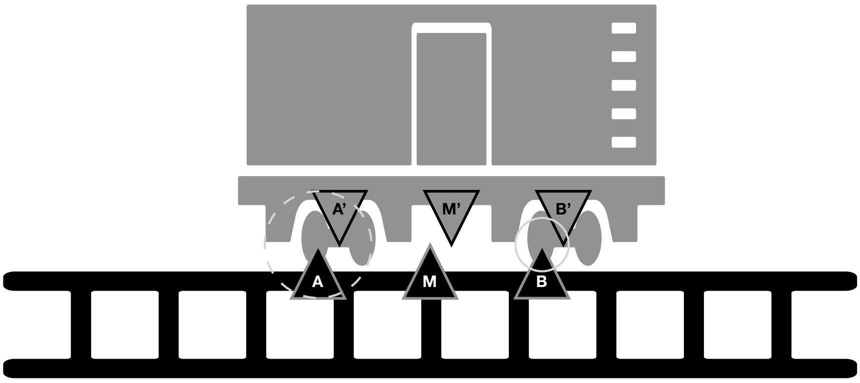

Figure 3.2

Then, clearly, in the train's track reference frame, the signals will arrive in the midpoint M in the following order:

1. the signal coming from A (figure 3.3)

and, after Δt,

2. the signal coming from B (figure 3.4)

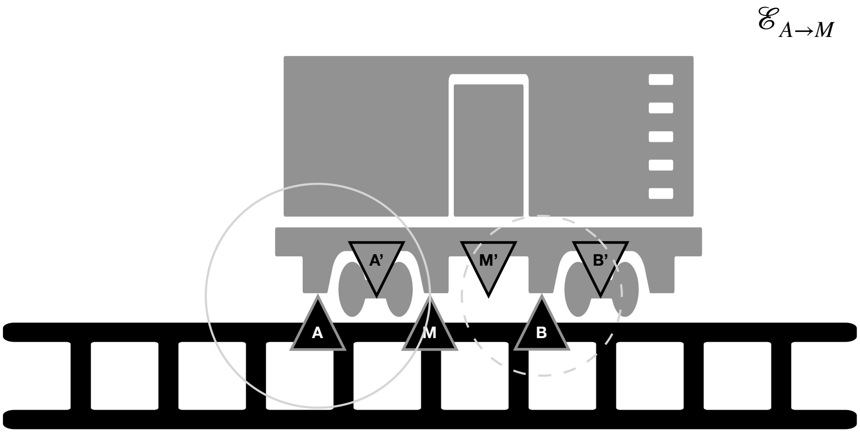

Figure 3.3

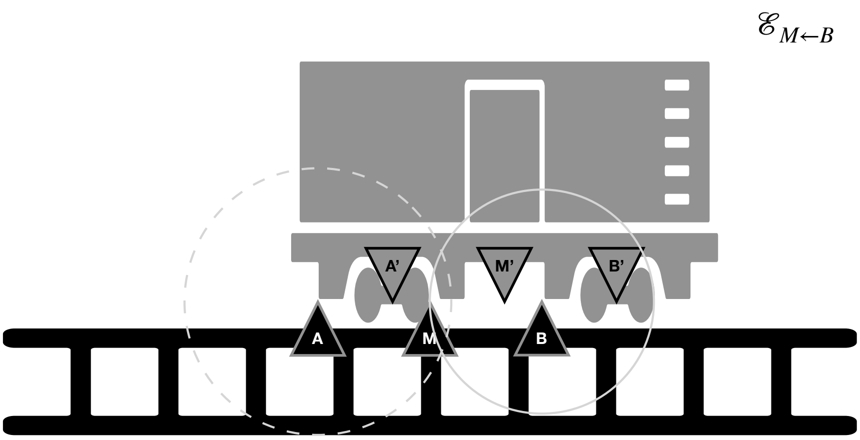

Figure 3.4

On the other hand, analyzing also in this case what happens in the train from the point of view of the track reference frame, we can conclude that the light beam coming from B' is anticipated of $\gamma^2 lv/c^2$ with respect to the beam emitted by A', as stated in section 2.2.3. Thus, if Δt above is such that $\Delta t < \gamma^2 lv/c^2$ then, in the train track reference frame, the signals emitted from A' and B' will arrive in the midpoint M' in the following order:

1. the signal coming from B (figure 3.5)

and, after $\gamma^2 lv/c^2 - \Delta t$,

2. the signal coming from A (figure 3.6)

The chronological order in the two references is obviously reversed.

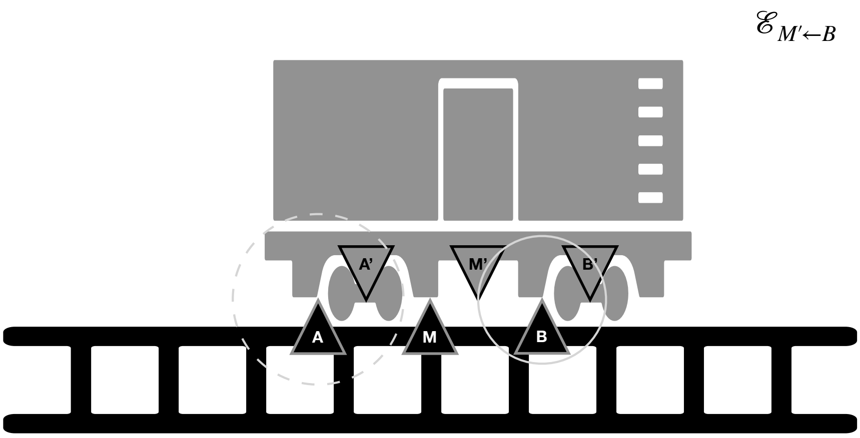

Figure 3.5

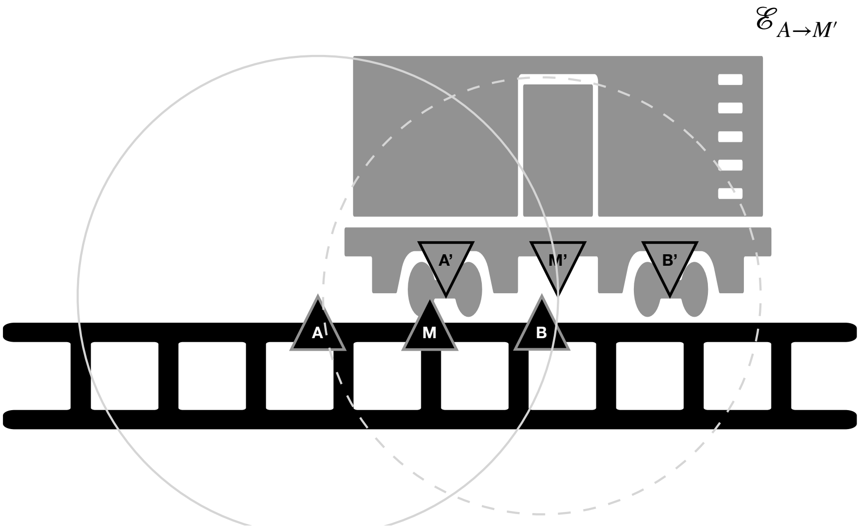

Figure 3.6

Note that not only the two light beams arrive at the midpoint M' reversed compared to the arrivals on M, in M''s reference frame M' sees itself struck the two events in reversed order as well. This tells us that the events arranged on the temporal line of an observer (the events that occur in the points of space that coincide with that observer) maintain the same temporal order in any frame of reference; on the other hand, events that occur in generic points of space-time can be arranged in an order that varies according to the chosen observer. From this we can conclude that, in general, both simultaneity and chronological order of events are dismissed.

3.2 Causality

If it is possible that times of events are actually reversible, what about the *principle of causality* so important to philosophers? Can causes and effects really exchange with each other? In special relativity cause and effects cannot be inverted: the only reversible events are the ones separated through space, as we saw in the previous paragraphs, since the mathematical component that allows such thing is vx/c^2. When we talk about causality though, we consider pairs of events that can be related in two different ways:

- Proper events of a certain medium, such that an event happening to that object, will later have a consequence on the same object.

- Events happening on a certain medium, that will later have a consequence on another medium.

The first case can be demonstrated as follows:

Proof. We consider two generic events $\mathcal{E}_1$ and $\mathcal{E}_2$ proper of a certain medium, and in that medium's timeline, they are ordered such that $t'_2 > t'_1$, where t'_2 is the proper time of $\mathcal{E}_2$ and t'_1 is the proper time of $\mathcal{E}_1$. Our goal it so prove that $t_2 > t_1$ as well, where t_2 and t_1 are the corresponding time coordinates of any frame. To achieve such result, will use again the Lorentz equation of time, where x_1 and x_2 are the spacial coordinates of the two events in any chosen frame. Note that the position of $\mathcal{E}_1$ and $\mathcal{E}_2$ coincide with the position of the medium, since the events are proper of it.

$$t'_2 > t'_1$$

$$\gamma\left(t_2 - \frac{vx_2}{c^2}\right) > \gamma\left(t_1 - \frac{vx_1}{c^2}\right)$$

since γ is always greater than 0, we are free to simplify it

$$t_2 - \frac{vx_2}{c^2} > t_1 - \frac{vx_1}{c^2}$$

$$t_2 > t_1 - \frac{vx_1}{c^2} + \frac{vx_2}{c^2}$$

$$t_2 > t_1 + \frac{v}{c^2}\left(x_2 - x_1\right)$$

considering that $\mathcal{E}_1$ and $\mathcal{E}_2$ coincide with the position of the medium, we can see that

$$x_2 = x_1 + v\Delta t = x_1 + v(t_2 - t_1)$$

substituting it inside the previous equation

$$t_2 > t_1 + \frac{v}{c^2}\left(x_1 + v(t_2 - t_1) - x_1\right)$$

$$t_2 > t_1 + \frac{v}{c^2}\left(v(t_2 - t_1)\right)$$

$$t_2 > t_1 + \frac{v^2}{c^2}\left(t_2 - t_1\right)$$

$$t_2 > t_1 + t_2\frac{v^2}{c^2} - t_1\frac{v^2}{c^2}$$

$$t_2 - t_2\frac{v^2}{c^2} > t_1 - t_1\frac{v^2}{c^2}$$

$$t_2\left(1 - \frac{v^2}{c^2}\right) > t_1\left(1 - \frac{v^2}{c^2}\right)$$

note that the term $\left(1 - \frac{v^2}{c^2}\right)$ is always greater than 0.

$$t_2 > t_1$$

$\square$

The second case's demonstration can be developed similarly to the first one:

Proof. Let $\mathcal{E}_1$ be a generic event of space-time and $\mathcal{E}_2$ it's consequence on a certain medium. In that medium's timeline, they are ordered such that $t'_2 > t'_1$, where t'_2 is the proper time of $\mathcal{E}_2$ and t'_1 is the non-proper time of $\mathcal{E}_1$. Again, x_1 and x_2 are the spacial coordinates of the two events in any chosen frame.

$$t'_2 > t'_1$$

$$\gamma\left(t_2 - \frac{vx_2}{c^2}\right) > \gamma\left(t_1 - \frac{vx_1}{c^2}\right)$$

$$t_2 - \frac{vx_2}{c^2} > t_1 - \frac{vx_1}{c^2}$$

$$t_2 > t_1 - \frac{vx_1}{c^2} + \frac{vx_2}{c^2}$$

$$t_2 > t_1 + \frac{v}{c^2}\left(x_2 - x_1\right)$$

in this case x_2 can be expressed as follows:

$$x_2 = x_1 + w\Delta t = x_1 + w(t_2 - t_1)$$

where w is the velocity with which $\mathcal{E}_1$ is interfering with the medium, it could be a signal or an emitted object, anything that would imply the consequence $\mathcal{E}_2$. Note that w can't be faster than c.

$$t_2 > t_1 + \frac{v}{c^2}\left(x_1 + w(t_2 - t_1) - x_1\right)$$

$$t_2 > t_1 + \frac{v}{c^2}\left(w(t_2 - t_1)\right)$$

$$t_2 > t_1 + \frac{vw}{c^2}\left(t_2 - t_1\right)$$

$$t_2 > t_1 + t_2\frac{vw}{c^2} - t_1\frac{vw}{c^2}$$

$$t_2 + t_2\frac{vw}{c^2} > t_1 - t_1\frac{vw}{c^2}$$

$$t_2\left(1 - \frac{vw}{c^2}\right) > t_1\left(1 - \frac{vw}{c^2}\right)$$

again, $\left(1 - \frac{vw}{c^2}\right)$ is always greater than 0, even if w get's close to $\pm c$

$$t_2 > t_1$$

□

Considering events separated throw space, that are not a consequence of each other, such as in 2.1.2, we can see that the time of the two events $t_{0,1} = 0$ and $t_{0,2} = 0$, in the train's reference frame become respectively

$$t'_{0,1} = \gamma\frac{dv}{c^2}$$

and

$$t'_{0,2} = -\gamma\frac{dv}{c^2}$$

The time of the two events shifted in a way such that they are equal, but with opposite signs. We could think that since $t'_{0,2}$ shifts time with a minus sign, than the moving observer actually went back in time, but this is not the case: the only thing that all frames of reference agree on is the occurrence of events, the shift of times is just a result of different alignments of the different observers.

Chapter 4

Conclusions

In conclusion, simultaneity is not absolute, any observer perceives events with an order and a separation related to his speed and distance from it's measurements.

The event's time variation is explained by equation 1.5, with particular attention to the member

$$\frac{vx}{c^2}$$

In Einstein's relativity, the simultaneity principle becomes relative to the velocity of the observers. In space-time all objects move carrying a different time, they all travel at the same speed (through space-time), but not in the same direction.

The orientation of space and time gives us different simultaneity of events, not just by an optical illusion, but in an actual difference in perspective and perception.

Two observers that move at different speeds have different definitions of space and time, which causes their space-time axes to be different, and perceive events differently, but the causality principle remains valid.

Relativity challenges your basic intuitions that you've built up from everyday experience. It says your experience of time is not what you think it is, that time is malleable. Your experience of space is not what you think it is; it can stretch and shrink.

-Brian Greene